Examples of proofs in axiomatic system E

Formal system E for Euclid's Elements.
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Theorem 1 (th_prop1aux2_01.) *Assuming that* $A \neq B$ *and* $on(A,p)$ *and* $on(B,p)$ *and* $AB \cong BC$ *and* $BC \cong CA$ *it holds that* $C \neq A$.

Proof:

1. From the fact $A \neq B$ it holds that $B \neq A$.
2. It holds that $A = C$ or $A \neq C$.
   3. Assume that: $A = C$.
      4. From the facts $BC \cong CA$ and $A = C$ it holds that $BA \cong AA$.
      5. From the fact $BA \cong AA$ it holds that $cong\_zero(B, A)$ (using ax_cong_zero1).
      6. From the fact $cong\_zero(B, A)$ it holds that $B = A$ (using $ax\_metric1\_1$).
      7. From the facts $B \neq A$ and $B = A$ we get contradiction.
   8. Assume that: $A \neq C$.
      9. From the fact $A \neq C$ it holds that $C \neq A$.
      10. The conclusion follows from the fact $C \neq A$.
11. The conjecture follows in all cases.

QED

Theorem 2 (th_prop1aux2_02.) *Assuming that* $A \neq B$ *and* $on(A,p)$ *and* $on(B,p)$ *and* $AB \cong BC$ *and* $BC \cong CA$ *and* $C \neq A$ *it holds that* $C \neq B$.

Proof:

1. From the fact $A \neq B$ it holds that $B \neq A$.
2. It holds that $B = C$ or $B \neq C$.
   3. Assume that: $B = C$.
      4. From the facts $BC \cong CA$ and $B = C$ it holds that $BB \cong BA$.
      5. From the fact $BB \cong BA$ it holds that $B = A$ (using ax_cong_eq1).
      6. From the facts $B \neq A$ and $B = A$ we get contradiction.
   7. Assume that: $B \neq C$.
      8. From the fact $B \neq C$ it holds that $C \neq B$.
      9. The conclusion follows from the fact $C \neq B$.
10. The conjecture follows in all cases.

QED

Theorem 3 (th_prop1aux2_03.) *Assuming that* $A \neq B$ *and* $on(A,p)$ *and* $on(B,p)$ *and* $AB \cong BC$ *and* $BC \cong CA$ *and* $C \neq A$ *and* $C \neq B$ *it holds that* $\neg on(C,p)$.

Proof:

1. It holds that $AB \cong BA$ (using $ax\_metric3$).
2. It holds that $BA \cong AB$ (using $ax\_metric3$).

4. From the facts $AB \cong BC$ and $AB \cong BA$ it holds that $BC \cong BA$ (using $ax\_cong\_transitivity$).
5. From the facts $BA \cong AB$ and $BA \cong AB$ it holds that $AB \cong AB$ (using $ax\_cong\_transitivity$).
6. From the facts $AB \cong BC$ and $AB \cong AB$ it holds that $BC \cong AB$ (using $ax\_cong\_transitivity$).
7. From the facts $BC \cong AB$ and $BC \cong CA$ it holds that $AB \cong CA$ (using $ax\_cong\_transitivity$).
8. From the facts $AB \cong CA$ and $AB \cong AB$ it holds that $CA \cong AB$ (using $ax\_cong\_transitivity$).
9. From the facts $CA \cong AB$ and $CA \cong AC$ it holds that $AB \cong AC$ (using $ax\_cong\_transitivity$).
10. From the facts $AB \cong AC$ and $AB \cong AB$ it holds that $AC \cong AB$ (using $ax\_cong\_transitivity$).
11. From the fact $A \neq B$ there exist a circle α, where A is the center of α and B is on circle α (using $ax\_lines\_and\_circles2$).
12. From the fact A is the center of α it holds that A is inside circle α (using $ax\_generalities3$).
13. From the facts A is the center of α and B is on circle α and $AC \cong AB$ it holds that C is on circle α (using $ax\_segment3\_1$).
14. From the fact $C \neq B$ it holds that $B \neq C$.
15. From the fact $A \neq B$ it holds that $B \neq A$.
16. It holds that $on(C, p)$ or $\neg on(C, p)$.
    17. Assume that: $on(C, p)$.
        18. From the facts $on(A, p)$ and $on(B, p)$ and $on(C, p)$ and A is inside circle α and B is on circle α and C is on circle α and $B \neq C$ it holds that $bet(B, A, C)$ (using $ax\_circle1$).
        19. From the fact $bet(B, A, C)$ it holds that $bet(C, A, B)$ and $B \neq A$ and $B \neq C$ and $\neg bet(A, B, C)$ (using ax_bet1).
        20. From the fact $bet(C, A, B)$ it holds that $bet(B, A, C)$ and $C \neq A$ and $C \neq B$ and $\neg bet(A, C, B)$ (using ax_bet1).
        21. From the fact $B \neq A$ there exist a circle β, where B is the center of β and A is on circle β (using $ax\_lines\_and\_circles2$).
        22. From the fact B is the center of β it holds that B is inside circle β (using $ax\_generalities3$).
        23. From the facts B is the center of β and A is on circle β and $BC \cong BA$ it holds that C is on circle β (using $ax\_segment3\_1$).
        24. From the fact $C \neq A$ it holds that $A \neq C$.
        25. From the facts $on(B, p)$ and $on(A, p)$ and $on(C, p)$ and B is inside circle β and A is on circle β and C is on circle β and $A \neq C$ it holds that $bet(A, B, C)$ (using $ax\_circle1$).
        26. From the facts $\neg bet(A, B, C)$ and $bet(A, B, C)$ we get contradiction.
    27. Assume that: $\neg on(C, p)$.
        28. The conclusion follows from the fact $\neg on(C, p)$.
29. The conjecture follows in all cases.

QED