High school geometry theorems

Hilbert's axiomatic system.
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Theorem 1 (th_7_01.) *Assuming that $A \neq B$ there exist line p, point C, such that $A \in p$ and $B \in p$ and $C \notin p$.*

Proof:

1. From the fact $A \neq B$ there exist a line p where $A \in p$ and $B \in p$ (using ax_I1).
2. There exist a point C and a point D and a point E where $\neg col(C, D, E)$ (using ax_I3b).
3. It holds that $C \in p$ or $C \notin p$.
   4. Assume that: $C \in p$.
      5. It holds that $D \in p$ or $D \notin p$.
         6. Assume that: $D \in p$.
            7. It holds that $E \in p$ or $E \notin p$.
               8. Assume that: $E \in p$.
                  9. From the facts $C \in p$ and $D \in p$ and $E \in p$ it holds that $col(C, D, E)$ (using ax_D1).
                  10. From the facts $\neg col(C, D, E)$ and $col(C, D, E)$ we get contradiction.
               11. Assume that: $E \notin p$.
                  12. The conclusion follows from the facts $A \in p$ and $B \in p$ and $E \notin p$.
            13. The conjecture follows in all cases.
         14. Assume that: $D \notin p$.
            15. The conclusion follows from the facts $A \in p$ and $B \in p$ and $D \notin p$.
      16. The conjecture follows in all cases.
   17. Assume that: $C \notin p$.
      18. The conclusion follows from the facts $A \in p$ and $B \in p$ and $C \notin p$.
19. The conjecture follows in all cases.

QED

Theorem 2 (th_7_02.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ there exist point D, such that $bet(B, C, D)$.*

Proof:

1. It holds that $B = C$ or $B \neq C$.
   2. Assume that: $B = C$.
      3. From the facts $C \notin p$ and $B = C$ it holds that $B \notin p$.
      4. From the facts $B \notin p$ and $B \in p$ we get contradiction.
   5. Assume that: $B \neq C$.
      6. From the fact $B \neq C$ there exist a point G where $bet(B, C, G)$ (using ax_II2).

7. The conclusion follows from the fact $bet(B,C,G)$.

8. The conjecture follows in all cases.

QED

Theorem 3 (th_7_03.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B,C,D)$ there exist point E, such that $bet(A,D,E)$.*

Proof:

1. From the fact $bet(B,C,D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B,C,D)$ and $bet(D,C,B)$ (using ax_II1).
2. From the fact $col(B,C,D)$ it holds that $col(B,D,C)$ and $col(C,B,D)$ and $col(C,D,B)$ and $col(D,B,C)$ and $col(D,C,B)$ (using ax_sym_col).
3. From the fact $A \neq B$ it holds that $B \neq A$.
4. From the facts $B \neq A$ and $B \in p$ and $A \in p$ and $C \notin p$ it holds that $\neg col(B,A,C)$ (using ax_D1a).
5. It holds that $A = D$ or $A \neq D$.
   6. Assume that: $A = D$.
      7. From the facts $\neg col(B,A,C)$ and $A = D$ it holds that $\neg col(B,D,C)$.
      8. From the facts $\neg col(B,D,C)$ and $col(B,D,C)$ we get contradiction.
   9. Assume that: $A \neq D$.
      10. From the fact $A \neq D$ there exist a point G where $bet(A,D,G)$ (using ax_II2).
      11. The conclusion follows from the fact $bet(A,D,G)$.
12. The conjecture follows in all cases.

QED

Theorem 4 (th_7_04.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B,C,D)$ and $bet(A,D,E)$ it holds that $\neg col(A,B,D)$.*

Proof:

1. From the fact $bet(B,C,D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B,C,D)$ and $bet(D,C,B)$ (using ax_II1).
2. From the fact $col(B,C,D)$ it holds that $col(B,D,C)$ and $col(C,B,D)$ and $col(C,D,B)$ and $col(D,B,C)$ and $col(D,C,B)$ (using ax_sym_col).
3. It holds that $D \in p$ or $D \notin p$.
   4. Assume that: $D \in p$.
      5. From the facts $B \neq D$ and $B \in p$ and $D \in p$ and $C \notin p$ it holds that $\neg col(B,D,C)$ (using ax_D1a).
      6. From the facts $\neg col(B,D,C)$ and $col(B,D,C)$ we get contradiction.
   7. Assume that: $D \notin p$.
      8. From the facts $A \neq B$ and $A \in p$ and $B \in p$ and $D \notin p$ it holds that $\neg col(A,B,D)$ (using ax_D1a).
      9. The conclusion follows from the fact $\neg col(A,B,D)$.
10. The conjecture follows in all cases.

QED

Theorem 5 (th_7_05.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B,C,D)$ and $bet(A,D,E)$ and $\neg col(A,B,D)$ there exist line q, such that $E \in q$ and $C \in q$.*

Proof:

1. It holds that $A = C$ or $A \neq C$.
   2. Assume that: $A = C$.
      3. From the facts $C \notin p$ and $A = C$ it holds that $A \notin p$.

4. From the facts $A \notin p$ and $A \in p$ we get contradiction.
5. Assume that: $A \neq C$.
6. From the fact $A \neq C$ there exist a line q where $A \in q$ and $C \in q$ (using ax_I1).
7. It holds that $B = C$ or $B \neq C$.
8. Assume that: $B = C$.
9. From the facts $C \notin p$ and $B = C$ it holds that $B \notin p$.
10. From the facts $B \notin p$ and $B \in p$ we get contradiction.
11. Assume that: $B \neq C$.
12. From the fact $B \neq C$ there exist a line r where $B \in r$ and $C \in r$ (using ax_I1).
13. It holds that $B = E$ or $B \neq E$.
14. Assume that: $B = E$.
15. From the facts $B \in r$ and $B = E$ it holds that $E \in r$.
16. The conclusion follows from the facts $E \in r$ and $C \in r$.
17. Assume that: $B \neq E$.
18. It holds that $C = E$ or $C \neq E$.
19. Assume that: $C = E$.
20. From the facts $C \in q$ and $C = E$ it holds that $E \in q$.
21. The conclusion follows from the facts $E \in q$ and $C \in q$.
22. Assume that: $C \neq E$.
23. From the fact $C \neq E$ there exist a line u where $C \in u$ and $E \in u$ (using ax_I1).
24. The conclusion follows from the facts $E \in u$ and $C \in u$.
25. The conjecture follows in all cases.
26. The conjecture follows in all cases.
27. The conjecture follows in all cases.
28. The conjecture follows in all cases.

QED

Theorem 6 (th_7_06.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ it holds that $A \notin q$.*

Proof:

1. From the fact $bet(A, D, E)$ it holds that $A \neq D$ and $A \neq E$ and $D \neq E$ and $col(A, D, E)$ and $bet(E, D, A)$ (using ax_II1).
2. From the fact $bet(B, C, D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B, C, D)$ and $bet(D, C, B)$ (using ax_II1).
3. From the fact $col(A, D, E)$ it holds that $col(A, E, D)$ and $col(D, A, E)$ and $col(D, E, A)$ and $col(E, A, D)$ and $col(E, D, A)$ (using ax_sym_col).
4. From the fact $col(B, C, D)$ it holds that $col(B, D, C)$ and $col(C, B, D)$ and $col(C, D, B)$ and $col(D, B, C)$ and $col(D, C, B)$ (using ax_sym_col).
5. From the facts $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ it holds that $\neg col(A, B, C)$ (using ax_D1a).
6. It holds that $A \in q$ or $A \notin q$.
7. Assume that: $A \in q$.
8. It holds that $B \in q$ or $B \notin q$.
9. Assume that: $B \in q$.
10. From the facts $A \in q$ and $B \in q$ and $C \in q$ it holds that $col(A, B, C)$ (using ax_D1).
11. From the facts $\neg col(A, B, C)$ and $col(A, B, C)$ we get contradiction.

12. Assume that: $B \notin q$.
13. It holds that $D \in q$ or $D \notin q$.
14. Assume that: $D \in q$.
15. From the facts $C \neq D$ and $C \in q$ and $D \in q$ and $B \notin q$ it holds that $\neg col(C, D, B)$ (using ax_D1a).
16. From the facts $\neg col(C, D, B)$ and $col(C, D, B)$ we get contradiction.
17. Assume that: $D \notin q$.
18. From the facts $A \neq E$ and $A \in q$ and $E \in q$ and $D \notin q$ it holds that $\neg col(A, E, D)$ (using ax_D1a).
19. From the facts $\neg col(A, E, D)$ and $col(A, E, D)$ we get contradiction.
20. The conjecture follows in all cases.
21. The conjecture follows in all cases.
22. Assume that: $A \notin q$.
23. The conclusion follows from the fact $A \notin q$.
24. The conjecture follows in all cases.

QED

Theorem 7 (th_7_07.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ and $A \notin q$ there exist plane α, such that $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$.*

Proof:

1. From the fact $\neg col(A, B, D)$ there exist a plane α, where $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ (using ax_I4a).
2. The conclusion follows from the facts $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$.

QED

Theorem 8 (th_7_08.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ and $A \notin q$ and $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ it holds that $C \in \alpha$.*

Proof:

1. From the fact $bet(B, C, D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B, C, D)$ and $bet(D, C, B)$ (using ax_II1).
2. From the fact $col(B, C, D)$ there exist a line s where $B \in s$ and $C \in s$ and $D \in s$ (using ax_D2).
3. From the facts $B \neq D$ and $B \in s$ and $D \in s$ and $B \in \alpha$ and $D \in \alpha$ it holds that $s \in \alpha$ (using ax_I6).
4. From the facts $s \in \alpha$ and $C \in s$ it holds that $C \in \alpha$ (using ax_D11).
5. The conclusion follows from the fact $C \in \alpha$.

QED

Theorem 9 (th_7_09.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ and $A \notin q$ and $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ and $C \in \alpha$ it holds that $E \in \alpha$.*

Proof:

1. From the fact $bet(A, D, E)$ it holds that $A \neq D$ and $A \neq E$ and $D \neq E$ and $col(A, D, E)$ and $bet(E, D, A)$ (using ax_II1).
2. From the fact $bet(B, C, D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B, C, D)$ and $bet(D, C, B)$ (using ax_II1).
3. From the fact $col(B, C, D)$ it holds that $col(B, D, C)$ and $col(C, B, D)$ and $col(C, D, B)$ and $col(D, B, C)$ and $col(D, C, B)$ (using ax_sym_col).

4. From the fact $A \neq D$ there exist a line s where $A \in s$ and $D \in s$ (using ax_I1).

5. From the facts $A \neq D$ and $A \in s$ and $D \in s$ and $A \in \alpha$ and $D \in \alpha$ it holds that $s \in \alpha$ (using ax_I6).

6. It holds that $B \in q$ or $B \notin q$.

 7. Assume that: $B \in q$.

  8. From the facts $B \neq C$ and $B \in q$ and $C \in q$ and $B \in \alpha$ and $C \in \alpha$ it holds that $q \in \alpha$ (using ax_I6).

  9. From the facts $q \in \alpha$ and $E \in q$ it holds that $E \in \alpha$ (using ax_D11).

  10. The conclusion follows from the fact $E \in \alpha$.

 11. Assume that: $B \notin q$.

  12. It holds that $A = C$ or $A \neq C$.

   13. Assume that: $A = C$.

    14. From the facts $col(C, B, D)$ and $A = C$ it holds that $col(A, B, D)$.

    15. From the facts $\neg col(A, B, D)$ and $col(A, B, D)$ we get contradiction.

   16. Assume that: $A \neq C$.

    17. From the fact $A \neq C$ there exist a line r where $A \in r$ and $C \in r$ (using ax_I1).

    18. From the facts $A \neq C$ and $A \in r$ and $C \in r$ and $A \in \alpha$ and $C \in \alpha$ it holds that $r \in \alpha$ (using ax_I6).

    19. It holds that $B \in r$ or $B \notin r$.

     20. Assume that: $B \in r$.

      21. It holds that $D \in r$ or $D \notin r$.

       22. Assume that: $D \in r$.

        23. It holds that $E \in r$ or $E \notin r$.

         24. Assume that: $E \in r$.

          25. From the facts $r \in \alpha$ and $E \in r$ it holds that $E \in \alpha$ (using ax_D11).

          26. The conclusion follows from the fact $E \in \alpha$.

         27. Assume that: $E \notin r$.

          28. From the facts $A \neq D$ and $A \in r$ and $D \in r$ and $E \notin r$ it holds that $\neg col(A, D, E)$ (using ax_D1a).

          29. From the facts $\neg col(A, D, E)$ and $col(A, D, E)$ we get contradiction.

        30. The conjecture follows in all cases.

       31. Assume that: $D \notin r$.

        32. From the facts $B \neq C$ and $B \in r$ and $C \in r$ and $D \notin r$ it holds that $\neg col(B, C, D)$ (using ax_D1a).

        33. From the facts $\neg col(B, C, D)$ and $col(B, C, D)$ we get contradiction.

      34. The conjecture follows in all cases.

     35. Assume that: $B \notin r$.

      36. It holds that $E \in s$ or $E \notin s$.

       37. Assume that: $E \in s$.

        38. From the facts $s \in \alpha$ and $E \in s$ it holds that $E \in \alpha$ (using ax_D11).

        39. The conclusion follows from the fact $E \in \alpha$.

       40. Assume that: $E \notin s$.

        41. From the facts $A \neq D$ and $A \in s$ and $D \in s$ and $E \notin s$ it holds that $\neg col(A, D, E)$ (using ax_D1a).

42. From the facts $\neg col(A, D, E)$ and $col(A, D, E)$ we get contradiction.
43. The conjecture follows in all cases.
44. The conjecture follows in all cases.
45. The conjecture follows in all cases.
46. The conjecture follows in all cases.

QED

Theorem 10 (th_7_10.) *Assuming that* $A \neq B$ *and* $A \in r$ *and* $B \in r$ *and* $D \notin r$ *and* $bet(B, D, E)$ *and* $bet(A, E, F)$ *and* $\neg col(A, B, E)$ *and* $F \in v$ *and* $D \in v$ *and* $A \notin v$ *and* $A \in \gamma2$ *and* $B \in \gamma2$ *and* $E \in \gamma2$ *and* $D \in \gamma2$ *and* $F \in \gamma2$ *it holds that* $v \in \gamma2$*.*

Theorem 11 (th_7_11.) *Assuming that* $A \neq B$ *and* $A \in r$ *and* $B \in r$ *and* $D \notin r$ *and* $bet(B, D, E)$ *and* $bet(A, E, F)$ *and* $\neg col(A, B, E)$ *and* $F \in v$ *and* $D \in v$ *and* $A \notin v$ *and* $A \in \gamma2$ *and* $B \in \gamma2$ *and* $E \in \gamma2$ *and* $D \in \gamma2$ *and* $F \in \gamma2$ *and* $v \in \gamma2$ *it holds that* $A \notin v$*.*

Theorem 12 (th_7_12.) *Assuming that* $A \neq B$ *and* $A \in p$ *and* $B \in p$ *and* $C \notin p$ *and* $bet(B, C, D)$ *and* $bet(A, D, E)$ *and* $\neg col(A, B, D)$ *and* $E \in q$ *and* $C \in q$ *and* $A \notin q$ *and* $A \in \alpha$ *and* $B \in \alpha$ *and* $D \in \alpha$ *and* $C \in \alpha$ *and* $E \in \alpha$ *and* $q \in \alpha$ *and* $A \notin q$ *it holds that* $B \notin q$*.*

Proof:

1. From the fact $bet(A, D, E)$ it holds that $A \neq D$ and $A \neq E$ and $D \neq E$ and $col(A, D, E)$ and $bet(E, D, A)$ (using ax_II1).
2. From the fact $bet(B, C, D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B, C, D)$ and $bet(D, C, B)$ (using ax_II1).
3. From the fact $bet(E, D, A)$ it holds that $E \neq D$ and $E \neq A$ and $D \neq A$ and $col(E, D, A)$ and $bet(A, D, E)$ (using ax_II1).
4. From the fact $D \neq E$ it holds that $E \neq D$.
5. It holds that $B \in q$ or $B \notin q$.
6. Assume that: $B \in q$.
7. It holds that $D \in q$ or $D \notin q$.
8. Assume that: $D \in q$.
9. From the facts $E \neq D$ and $E \in q$ and $D \in q$ and $A \notin q$ it holds that $\neg col(E, D, A)$ (using ax_D1a).
10. From the facts $\neg col(E, D, A)$ and $col(E, D, A)$ we get contradiction.
11. Assume that: $D \notin q$.
12. From the facts $B \neq C$ and $B \in q$ and $C \in q$ and $D \notin q$ it holds that $\neg col(B, C, D)$ (using ax_D1a).
13. From the facts $\neg col(B, C, D)$ and $col(B, C, D)$ we get contradiction.
14. The conjecture follows in all cases.
15. Assume that: $B \notin q$.
16. The conclusion follows from the fact $B \notin q$.
17. The conjecture follows in all cases.

QED

Theorem 13 (th_7_13.) *Assuming that* $A \neq B$ *and* $A \in p$ *and* $B \in p$ *and* $C \notin p$ *and* $bet(B, C, D)$ *and* $bet(A, D, E)$ *and* $\neg col(A, B, D)$ *and* $E \in q$ *and* $C \in q$ *and* $A \notin q$ *and* $A \in \alpha$ *and* $B \in \alpha$ *and* $D \in \alpha$ *and* $C \in \alpha$ *and* $E \in \alpha$ *and* $q \in \alpha$ *and* $A \notin q$ *and* $B \notin q$ *it holds that* $D \notin q$*.*

Proof:

1. From the fact $bet(B, C, D)$ it holds that $B \neq C$ and $B \neq D$ and $C \neq D$ and $col(B, C, D)$ and $bet(D, C, B)$ (using ax_II1).
2. From the fact $bet(D, C, B)$ it holds that $D \neq C$ and $D \neq B$ and $C \neq B$ and $col(D, C, B)$ and $bet(B, C, D)$ (using ax_II1).
3. From the fact $C \neq D$ it holds that $D \neq C$.
4. It holds that $D \in q$ or $D \notin q$.
   5. Assume that: $D \in q$.
      6. From the facts $D \neq C$ and $D \in q$ and $C \in q$ and $B \notin q$ it holds that $\neg col(D, C, B)$ (using ax_D1a).
      7. From the facts $\neg col(D, C, B)$ and $col(D, C, B)$ we get contradiction.
   8. Assume that: $D \notin q$.
      9. The conclusion follows from the fact $D \notin q$.
10. The conjecture follows in all cases.

QED

Theorem 14 (th_7_14.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ and $A \notin q$ and $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ and $C \in \alpha$ and $E \in \alpha$ and $q \in \alpha$ and $A \notin q$ and $B \notin q$ and $D \notin q$ it holds that line q intersects segment BD.*

Proof:

1. From the facts $B \notin q$ and $D \notin q$ and $C \in q$ and $bet(B, C, D)$ it holds that line q intersects segment BD (using ax_cut_1).
2. The conclusion follows from the fact line q intersects segment BD.

QED

Theorem 15 (th_7_15.) *Assuming that $A \neq B$ and $A \in p$ and $B \in p$ and $C \notin p$ and $bet(B, C, D)$ and $bet(A, D, E)$ and $\neg col(A, B, D)$ and $E \in q$ and $C \in q$ and $A \notin q$ and $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ and $C \in \alpha$ and $E \in \alpha$ and $q \in \alpha$ and $A \notin q$ and $B \notin q$ and $D \notin q$ and line q intersects segment BD there exist line r, such that $pash(A, B, D, q, \alpha)$ and $D \neq A$ and $D \in r$ and $A \in r$.*

Proof:

1. From the facts $A \in p$ and $B \in p$ and $A \in p$ it holds that $col(A, B, A)$ (using ax_D1).
2. From the facts $\neg col(A, B, D)$ and $A \in \alpha$ and $B \in \alpha$ and $D \in \alpha$ and $q \in \alpha$ and $A \notin q$ and $B \notin q$ and $D \notin q$ and line q intersects segment BD it holds that pash(A, B, D, q, α) (using ax_pash_1).
3. It holds that $A = D$ or $A \neq D$.
   4. Assume that: $A = D$.
      5. From the facts $col(A, B, A)$ and $A = D$ it holds that $col(A, B, D)$.
      6. From the facts $\neg col(A, B, D)$ and $col(A, B, D)$ we get contradiction.
   7. Assume that: $A \neq D$.
      8. From the fact $A \neq D$ there exist a line r where $A \in r$ and $D \in r$ (using ax_I1).
      9. From the fact $A \neq D$ it holds that $D \neq A$.
      10. The conclusion follows from the facts pash(A, B, D, q, α) and $D \neq A$ and $D \in r$ and $A \in r$.
11. The conjecture follows in all cases.

QED

Theorem 16 (th_7_16.) *Assuming that $A \neq B$ and $A \in r$ and $B \in r$ and $D \notin r$ and $bet(B, D, E)$ and $bet(A, E, F)$ and $\neg col(A, B, E)$ and $F \in v$ and $D \in v$ and $A \notin v$ and $A \in \gamma 2$ and $B \in \gamma 2$ and $E \in \gamma 2$ and $D \in \gamma 2$ and $F \in \gamma 2$ and $v \in \gamma 2$ and $A \notin v$ and $B \notin v$ and $E \notin v$ and line v intersects segment BE and $pash(A, B, E, v, \gamma 2)$ and $E \neq A$ and $E \in q1$ and $A \in q1$ there exist point K, such that $bet(A, K, B)$.*
