Alldifferent constraint solver in SMT

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Outline

1. Introduction to alldifferent constraint
2. The alldifferent theory
3. Our alldifferent *SMT* solver
4. Experimental evaluation
5. Future work and conclusions

Outline

Definition of alldifferent constraint

Definition

Given a set of variables $x_1, x_2, \dots, x_n$, where each variable x_i takes values from its corresponding finite domain $D(x_i)$, then $\mathit{alldiff}\,(x_1, x_2, \dots, x_n)$ means that every two different variables must take different values ($i \neq j \Rightarrow x_i \neq x_j$).

Applications

Broad variety of problems can be reduced to alldifferent:

- Puzzle solving (Sudoku, Latin Square, Eight Queens).
- Scheduling and timetabling.

Example of alldifferent based problem

Example (Latin square 5×5)

	3	4		
3	4	5		
4	5			
5				

Notes

- Each cell should be filled in with a value from 1 to 5.
- Each row and each column is constrained by alldifferent constraint.
- Some values are already given.

Example of alldifferent based problem

Example (Latin square 5×5)

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Notes

- Each cell should be filled in with a value from 1 to 5.
- Each row and each column is constrained by alldifferent constraint.
- Some values are already given.

Representation of problem

$$
\begin{gathered}
D(x_{11}) = \{1, \ldots, 5\} \\
D(x_{12}) = \{1, \ldots, 5\} \\
\ldots \\
D(x_{55}) = \{1, \ldots, 5\} \\
alldiff(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}) \\
alldiff(x_{21}, x_{22}, x_{23}, x_{24}, x_{25}) \\
\ldots \\
alldiff(x_{15}, x_{25}, x_{35}, x_{45}, x_{55}) \\
x_{51} = 5 \\
x_{41} = 4 \\
\ldots \\
x_{23} = 4
\end{gathered}
$$

Notes

- For each cell one variable is introduced.
- For each variable, its domain consists of values $\{1, \ldots, 5\}$.
- For each row and each column – one alldifferent constraint.
- For each given value – one equality constraint.

Representation of solution

$$x_{11} = 1, x_{12} = 2, \ldots, x_{15} = 5$$
$$x_{21} = 2, x_{22} = 3, \ldots, x_{25} = 1$$
$$\ldots$$
$$x_{51} = 5, \ldots, x_{52} = 1, x_{55} = 4$$

Notes

Solution of the problem is an assignment of values to variables satisfying all the constraints

Related work

Related work

- Algorithm implemented within CP solvers
- Arc-consistency for alldifferent – (Régin 1994)
- Explanation generating procedures in CP (Gent et al, 2010)
- Infinite and large domains (Quimper and Walsh, 2004)
- Bitvector alldifferent consistency checking (Biere and Brummayer, 2008)
- Usage of alldifferent within SMT (Nieuwenhuis et al, 2007)

What is our goal?

The goal is to...

- express a single alldifferent constraint as a first order theory
- construct a $DPLL(T)$-compliant theory solver (AD-solver) for such theory
- combine multiple AD-solvers to solve problems with multiple alldifferent constraints (Delayed Theory Combination)

Outline

The alldifferent theory

Signature and axioms

- The signature consists of constant symbols $\overline{x}_i$ for each variable x_i and constant symbols $\overline{d}_j$ for each value $d_j \in \bigcup_i D(x_i)$
- Axioms of sanity: $\overline{d}_i \neq \overline{d}_j$, for $i \neq j$ (different constants represent different values)
- Domain axioms: $\bigvee_{d \in D(x)} \overline{x} = \overline{d}$ for each variable x (x should take value from its domain)
- Axioms of difference: $\overline{x}_i \neq \overline{x}_j$ for $i \neq j$ (ensure alldifferent is satisfied)

Outline

AD-solver functionality:

- Conflict detection (based on optimal matchings)
- Theory propagation (based on Régin's algorithm)
- Propagation and conflict explaining (based on Minimal Obstacle Sets (MOS), contributed)

Matching problem and alldifferent

AD-solver

- *AD*-solver is based on the matching problem in bipartite graphs
- A bipartite graph is assigned to the alldifferent constraint (called its value graph)

Example of value graph

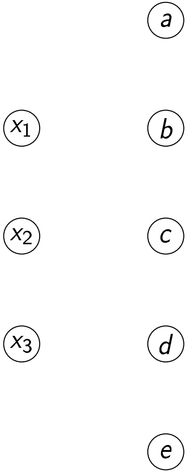
Example

$D(x_1) = \{a, c, d\}$, $D(x_2) = \{b, d, e\}$,
$D(x_3) = \{a, b\}$, $alldiff(x_1, x_2, x_3)$

Notes

- Each vertex at the left side corresponds to one variable.

Example of value graph



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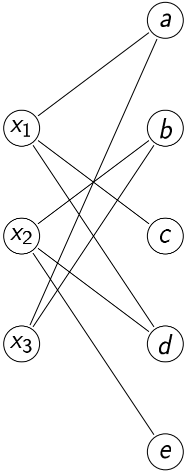
- Each vertex at the right side corresponds to one value.

Example of value graph



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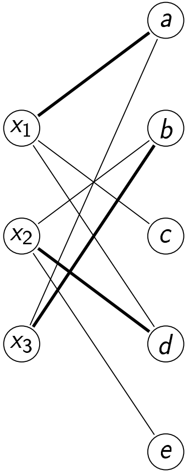
- Each vertex at the left side corresponds to one variable.
- Each vertex at the right side corresponds to one value.
- Each variable is connected to values from its domain.

Example of value graph



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Notes

- Each vertex at the left side corresponds to one variable.
- Each vertex at the right side corresponds to one value.
- Each variable is connected to values from its domain.
- Solution corresponds to matching that covers left side vertices.

Matching problem and alldifferent

Important concepts

- Alternating path – path containing edges that alternatively belong to the current matching
- Augmenting path – alternating path connecting unmatched vertices at opposite sides
- Directed value graph – the value graph with edges oriented from left to right if they belong to the matching, and from right to left otherwise

Conflict detection

Conflict detection in alldifferent

- Optimal matchings – matchings with maximal cardinality
- alldifferent is satisfiable iff an optimal matching covers left side vertices
- An optimal matching construction – augmenting paths based algorithms (e.g. Hopcroft&Karp)
- The current matching is incrementally augmented using found augmenting paths until an optimal matching is reached (incremental property)

Theory propagations

Classification of edges in the value graph

- Vital edge – belongs to all optimal matchings (equality propagated)
- Inconsistent edge – doesn't belong to any optimal matching (disequality propagated)
- Alternating edge – neither vital nor inconsistent

Theory propagations

Régin's algorithm

- Searching for alternating edges:
  - search for edges reachable from unmatched vertices (BFS, using alternating paths)
  - search for edges that belong to alternating cycles (Tarjan's algorithm for strongly connected components)
- Non-alternating (vital or inconsistent) edges cause propagations

Conflict and propagations explaining

Algorithm for finding minimal explanation

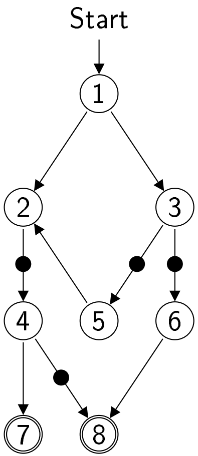
- The explanation generating procedure is the main contribution of our work
- The algorithm we propose finds explanations that are minimal in sense of inclusion.
- The problem of propagation and conflict explaining is reduced to the Minimal Obstacle Set Problem (MOS), also introduced and solved in our work

Conflict and propagations explaining

MOS problem

- MOS problem is the problem of finding subcut of a given cut in a directed graph that is minimal in sense of inclusion
- Edges belonging to the cut are called obstacles in our terminology (hence the name of the problem)
- Efficiency: the algorithm executes in one graph traversal

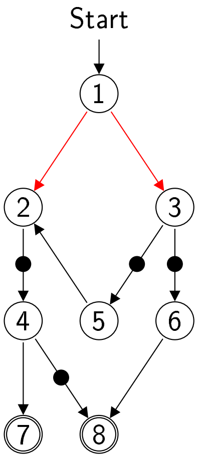
Conflict and propagations explaining



Notes

- Start vertex 1, final vertices $\{7, 8\}$.

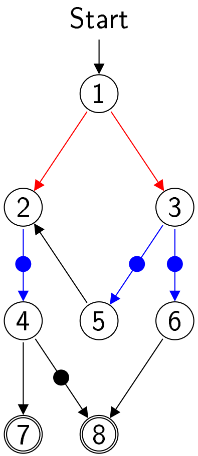
Conflict and propagations explaining



Notes

- Start vertex 1, final vertices $\{7, 8\}$.
- Check for reachable obstacles.

Conflict and propagations explaining



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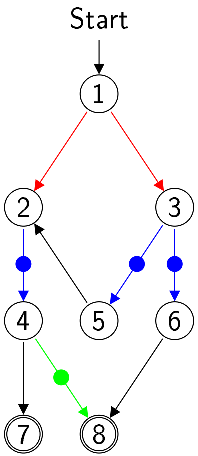

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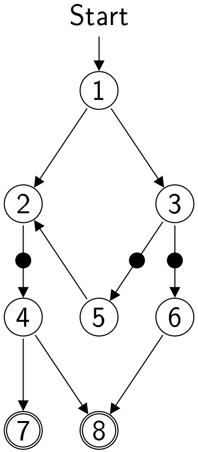
- Start vertex 1, final vertices $\{7, 8\}$.
- Check for reachable obstacles.
- Obstacle $(4, 8)$ is unreachable. It may be released.

Conflict and propagations explaining



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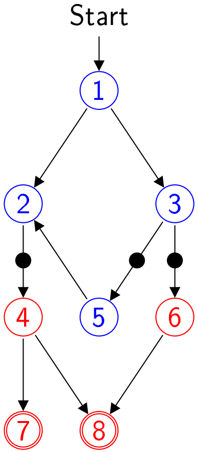

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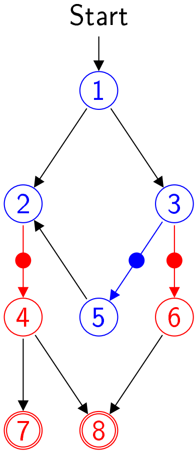
- Start vertex 1, final vertices $\{7, 8\}$.
- Check for reachable obstacles.
- Obstacle $(4, 8)$ is unreachable. It may be released.
- Blue vertices are blocked. Red vertices are unblocked.

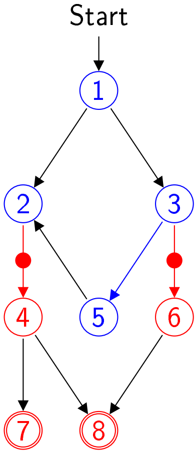
Conflict and propagations explaining



Notes

- Start vertex 1, final vertices $\{7, 8\}$.
- Check for reachable obstacles.
- Obstacle $(4, 8)$ is unreachable. It may be released.
- Blue vertices are blocked. Red vertices are unblocked.
- Obstacle $(3, 5)$ may be released, because 3 and 5 are both blocked.

Conflict and propagations explaining



Notes

- Start vertex 1, final vertices $\{7, 8\}$.
- Check for reachable obstacles.
- Obstacle $(4, 8)$ is unreachable. It may be released.
- Blue vertices are blocked. Red vertices are unblocked.
- Obstacle $(3, 5)$ may be released, because 3 and 5 are both blocked.
- Minimal set of obstacles: $(2, 4)$ and $(3, 6)$.

Conflict and propagations explaining

Reduction to MOS problem

- Obstacle set: edges removed from the value graph due to literals in the partial model
- Conflict explaining: Non-optimal matching cannot be augmented – unmatched vertices at the opposite sides are separated by the obstacles
- Propagation explaining: Edges became non-alternating – they are separated from the unmatched vertices by the obstacles (and/or alternating cycles containing them are broken by the obstacles)

Outline

Experimental evaluation

Experiments

- Sudoku instances 25×25, randomly generated with around 40% cells filled in
- 200 instances, time limit: 2 minutes per instance
- Compared solvers: argoalldiff (with and without explanations), argosat, minion, yices (EUF encoding, SMT-LIB's distinct)

Experimental evaluation

Solver	**Solved instances**	**Average time on solved instances**
argoalldiff (with expl.)	194	8.8s
minion	174	10s
argosat	172	18s
argoalldiff (without expl.)	169	18s
yices (using `distinct`)	0	-
yices (using EUF)	0	-

Experimental evaluation

Experimental evaluation

- Average number of conflicts: 460 (argoalldiff) vs 33000 (argosat)
- Reduction of explanations size: 27% for theory propagations, 9% for conflicts
- Explanation procedures time consumption: only 3% of the overall execution time

Outline

Future work

Future work

- Efficiency improvement of our prototypical implementation
- Possible application: timetabling (teaching timetable for Faculty of Mathematics)

Conclusions

Conclusions

- Alldifferent constraint: expressed as a first order theory
- *SMT*-approach: reducing alldifferent based problems to *SMT*.
- Efficient theory solver based on the matching problem.
- Conflict and propagation explaining: new efficient algorithm is proposed.

THANK YOU :)